

L^p -SOLVABILITY OF NONLOCAL PARABOLIC EQUATIONS WITH SPATIAL DEPENDENT AND NON-SMOOTH KERNELS*

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ABSTRACT. In this paper we prove the optimal L^p -solvability of nonlocal parabolic equation with spatial dependent and non-smooth kernels.

1. INTRODUCTION

In this paper we are considering the L^p -estimate of the following nonlocal operator:

$$\mathcal{L}^a f = \int_{\mathbb{R}^d} [f(x+y) - f(x) - y^{(\alpha)} \cdot \nabla f(x)] a(x, y) |y|^{-d-\alpha} dy, \quad (1)$$

where $\alpha \in (0, 2)$, $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ is a measurable function and

$$y^{(\alpha)} := 1_{\alpha \in (1, 2)} y + 1_{\alpha=1} y 1_{|y| \leq 1}.$$

When $a(x, y)$ is *smooth* and *0-homogenous* in y , or $a(x, y) = a(y)$ is *independent* of x , the L^p -estimates for this type of operators have been studied by Mikulevicius-Pragarauskas [13] and Dong-Kim [8] (see also [19]). However, for nonlinear applications, the smoothness and spatial-independence assumptions are usually not satisfied.

Let us now look at a nonlinear example. Consider the following variational integral appeared in nonlocal image and signal processing [9]:

$$V(\theta) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(\theta(x) - \theta(y)) \kappa(x - y) |y - x|^{-d-\alpha} dx dy, \quad \alpha \in (0, 2),$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ is an even convex C^2 -function and $\kappa(-x) = \kappa(x)$. Assume that ϕ and κ satisfy that for some $\Lambda > 0$,

$$\phi(0) = 0, \quad \Lambda^{-1} \leq \phi''(x) \leq \Lambda,$$

and

$$\Lambda^{-1} \leq \kappa(x) \leq \Lambda.$$

The Euler-Lagrange equation corresponding to $V(\theta)$ is given by

$$\int_{\mathbb{R}^d} \phi'(\theta(t, y) - \theta(t, x)) \kappa(y - x) |y - x|^{-d-\alpha} dy = 0.$$

In [6], Caffarelli, Chan and Vasseur firstly considered the following time dependence problem:

$$\partial_t \theta(t, x) = \int_{\mathbb{R}^d} \phi'(\theta(y) - \theta(x)) \kappa(y - x) |y - x|^{-d-\alpha} dy,$$

and proved that for any $\theta_0 \in \mathbb{H}^{1,2}$, there exists a unique global classical $C^{1,\beta}$ -solution to the above equation with $\theta(0, \cdot) = \theta_0$ in the L^2 -sense. The existence of weak solutions with non-increasing

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energy can be deduced by the standard energy argument. To address the regularity problem, they followed the classical idea of De Giorgi and considered the following linearized equation

$$\partial_t w(t, x) = \int_{\mathbb{R}^d} \phi''(\theta(t, y) - \theta(t, x))(w(t, y) - w(t, x))\kappa(y - x)|y - x|^{-d-\alpha} dy, \quad (2)$$

where $w(t, x) = \nabla \theta(t, x)$. If we set

$$\hat{k}(t, x, y) = \phi''(\theta(t, y) - \theta(t, x))\kappa(y - x) = \phi''\left((y - x) \cdot \int_0^1 w(t, x + s(y - x)) ds\right)\kappa(y - x),$$

then, since ϕ'' is an even function, we have

$$\hat{k}(t, x, y) = \hat{k}(t, y, x),$$

and equation (2) is understood in the weak sense: for all $\eta \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \partial_t w(t, x) \eta(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (w(t, y) - w(t, x))(\eta(y) - \eta(x))\kappa(t, x, y)|y - x|^{-d-\alpha} dy dx.$$

Clearly, if we let

$$a(t, x, y) := \hat{k}(t, x, x + y),$$

then equation (2) becomes

$$\partial_t w(t, x) = \int_{\mathbb{R}^d} (w(t, x + y) - w(t, x))a(t, x, y)|y|^{-d-\alpha} dy.$$

Notice that $a(t, x, y)$ is usually not smooth apriori in x and y . This type of equation is our main motivation.

This paper is organized as follows: In Section 2, we give some necessary spaces. In Section 3, we prove some estimates of nonlocal integral operators. In Section 4, the linear nonlocal parabolic equation is studied. In a forthcoming paper, we shall use the result obtained in this paper to study the stochastic differential equations with spatial dependence jump-diffusion coefficients (cf. [18]).

CONVENTION: Throughout this paper, we shall use C with or without subscripts to denote an unimportant constant.

2. PRELIMINARIES

In this section we introduce some necessary spaces of Dini-type (cf. [15, p.30, (25)]). Let \mathcal{A}_0 be the space of all real bounded measurable functions $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ with finite norm

$$\|a\|_{\mathcal{A}_0} := \sup_{x, y \in \mathbb{R}^d} |a(x, y)| + \int_0^1 \frac{\omega_a^{(0)}(r)}{r} dr < +\infty,$$

where

$$\omega_a^{(0)}(r) := \sup_{x \in \mathbb{R}^d} \sup_{|y| \leq r} |a(x, y) - a(x, 0)|. \quad (3)$$

Let $\mathcal{A}_1 \subset \mathcal{A}_0$ be the subspace with finite norm

$$\|a\|_{\mathcal{A}_1} := \|a\|_{\mathcal{A}_0} + \int_0^1 \frac{\omega_a^{(1)}(r)}{r} dr < +\infty,$$

where

$$\omega_a^{(1)}(r) := \sup_{|x - x'| \leq r} |a(x, 0) - a(x', 0)|. \quad (4)$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $p > 1$ and $\beta \geq 0$, let $\mathbb{H}^{\beta,p} := (I - \Delta)^{-\frac{\beta}{2}}(L^p)$ be the Bessel potential space with the norm

$$\|f\|_{\mathbb{H}^{\beta,p}} := \|(I - \Delta)^{\frac{\beta}{2}} f\|_p \sim \|f\|_p + \|(-\Delta)^{\frac{\beta}{2}} f\|_p,$$

and for $q \in [1, \infty]$, let $\mathbb{B}_q^{\beta,p}$ be the Besov space defined by

$$\mathbb{B}_q^{\beta,p} := (L^p, \mathbb{H}^{k,p})_{\frac{\beta}{k}, q},$$

where $k \in \mathbb{N}$ and $\beta < k$, and $(\cdot, \cdot)_{\frac{\beta}{k}, p}$ stands for the real interpolation space. Let us write

$$\mathbb{W}^{\beta,p} := \mathbb{B}_p^{\beta,p}.$$

It is well-known that if β is an integer and $p > 1$, an equivalent norm in $\mathbb{W}^{\beta,p} = \mathbb{H}^{\beta,p}$ is given by

$$\|f\|_{\mathbb{W}^{\beta,p}} := \sum_{k=0}^{\beta} \|\nabla^k f\|_p,$$

where ∇^k denotes the k -order generalized gradient; and if $0 < \beta \neq \text{integer}$ and $p > 1$, an equivalent norm in $\mathbb{W}^{\beta,p}$ is given by

$$\|f\|_{\mathbb{W}^{\beta,p}} := \|f\|_p + \sum_{k=0}^{[\beta]} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla^k f(x) - \nabla^k f(y)|^p}{|x - y|^{d + \{\beta\}p}} dx dy \right)^{\frac{1}{p}}, \quad (5)$$

where for a number $\beta > 0$, $[\beta]$ denotes the integer part of β and $\{\beta\} := \beta - [\beta]$. It is also well-known that Riesz's transform $\nabla(-\Delta)^{-\frac{1}{2}}$ is a bounded linear operator in L^p -space for any $p > 1$ (see [15]). Moreover, the following interpolation inequality holds: for any $\beta \in (0, \gamma)$, $p > 1$ and $f \in \mathbb{H}^{\gamma,p}$,

$$\|(-\Delta)^{\frac{\beta}{2}} f\|_p \leq C \|f\|_p^{1 - \frac{\beta}{\gamma}} \|(-\Delta)^{\frac{\gamma}{2}} f\|_p^{\frac{\beta}{\gamma}}. \quad (6)$$

The following lemma is an easy consequence of [11, Lemma 2.1].

Lemma 2.1. *For any $\beta \in (0, 1)$, there exists a constant $C = C(\beta, d) > 0$ such that for all $p \geq 1$ and $f \in \mathbb{H}^{\beta,p}$,*

$$\|f(\cdot + y) - f(\cdot)\|_p \leq C |y|^\beta \|(-\Delta)^{\frac{\beta}{2}} f\|_p. \quad (7)$$

For each $t \in [0, 1]$, write $\mathbb{Y}_t^{\beta,p} := L^p([0, t]; \mathbb{H}^{\beta,p})$ with the norm

$$\|u\|_{\mathbb{Y}_t^{\beta,p}} := \left(\int_0^t \|u(s)\|_{\mathbb{H}^{\beta,p}}^p ds \right)^{\frac{1}{p}},$$

and let $\mathbb{X}_t^{\beta,p}$ be the completion of all functions $u \in C^\infty([0, t]; \mathcal{S}(\mathbb{R}^d))$ with respect to the norm

$$\|u\|_{\mathbb{X}_t^{\beta,p}} := \sup_{s \in [0, t]} \|u(s)\|_{\mathbb{H}^{\beta-1,p}} + \|u\|_{\mathbb{Y}_t^{\beta,p}} + \|\partial_t u\|_{\mathbb{Y}_t^{\beta-1,p}}.$$

It is well-known that (cf. [1, p.180, Theorem III 4.10.2]),

$$\mathbb{X}_t^{\beta,p} \hookrightarrow C([0, t]; \mathbb{W}^{\beta - \frac{1}{p}, p}). \quad (8)$$

For simplicity of notation, we also write

$$\mathbb{X}^{\beta,p} := \mathbb{X}_1^{\beta,p}, \quad \mathbb{Y}^{\beta,p} := \mathbb{Y}_1^{\beta,p}.$$

3. L^p -ESTIMATE OF NONLOCAL OPERATORS

Let ν be a σ -finite measure on \mathbb{R}^d , which is called a Lévy measure if $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} 1 \wedge |x|^2 \nu(dx) < +\infty.$$

Let Σ be a finite measure on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d . For $\alpha \in (0, 2)$, define

$$\nu^{(\alpha)}(B) := \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty \frac{1_B(r\theta) dr}{r^{1+\alpha}} \right) \Sigma(d\theta), \quad B \in \mathcal{B}(\mathbb{R}^d). \quad (9)$$

Then $\nu^{(\alpha)}$ is the Lévy measure corresponding to the α -stable process.

Definition 3.1. (i) Let ν_1 and ν_2 be two Borel measures on \mathbb{R}^d . We say that ν_1 is less than ν_2 if

$$\nu_1(B) \leq \nu_2(B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

and we simply write $\nu_1 \leq \nu_2$ in this case.

(ii) The Lévy measure $\nu^{(\alpha)}$ defined by (9) is called nondegenerate if

$$\int_{\mathbb{S}^{d-1}} |\theta_0 \cdot \theta|^\alpha \Sigma(d\theta) \neq 0, \quad \forall \theta_0 \in \mathbb{S}^{d-1}. \quad (10)$$

Throughout this paper we make the following assumption:

(H $^{(\alpha)}_\nu$) Let ν be a Lévy measure and satisfy that for some $\alpha \in (0, 2)$,

$$\nu_1^{(\alpha)} \leq \nu \leq \nu_2^{(\alpha)}, \quad 1_{\alpha=1} \int_{r < |x| < R} y \nu(dy) = 0, \quad 0 < r < R < +\infty, \quad (11)$$

where $\nu_i^{(\alpha)}$, $i = 1, 2$ are two Lévy measures with the form (9), and $\nu_1^{(\alpha)}$ is nondegenerate.

Let us recall the following result from [19, Corollary 4.4].

Theorem 3.2. Assume **(H $^{(\alpha)}_\nu$)** with $\alpha \in (0, 2)$. Then for any $p \in (1, \infty)$, there exists a constant $C_0 \in (0, 1)$ such that for all $f \in \mathbb{H}^{\alpha, p}$,

$$C_0 \|(-\Delta)^{\frac{\alpha}{2}} f\|_p \leq \|\mathcal{L}^\nu f\|_p \leq C_0^{-1} \|(-\Delta)^{\frac{\alpha}{2}} f\|_p. \quad (12)$$

Below, for simplicity of notation, we write

$$\mathcal{J}_f^{(\alpha)}(x, y) := f(x + y) - f(x) - y^{(\alpha)} \cdot \nabla f(x). \quad (13)$$

We first prepare the following lemma for later use.

Lemma 3.3. Suppose that $a \in \mathcal{A}_0$ and $\nu \leq \nu^{(\alpha)}$ for some $\alpha \in (0, 2)$. For any $p > 1$, there exists a constant $C = C(\alpha, p, d) > 0$ such that for all $f \in \mathbb{H}^{\alpha, p}$ and $\varepsilon \in (0, 1)$,

$$\left\| \int_{|y| \leq \varepsilon} \mathcal{J}_f^{(\alpha)}(\cdot, y) (a(\cdot, y) - a(\cdot, 0)) \nu(dy) \right\|_p \leq C \|(-\Delta)^{\frac{\alpha}{2}} f\|_p \int_0^\varepsilon \frac{\omega_a^{(0)}(r)}{r} dr,$$

where $\omega_a^{(0)}$ is defined by (3).

Proof. Let us look at the case of $\alpha \in [1, 2)$. Since $a \in \mathcal{A}_0$, by Minkowski's inequality we have

$$\begin{aligned} & \left\| \int_{|y| \leq \varepsilon} [f(\cdot + y) - f(\cdot) - y \cdot \nabla f(\cdot)] (a(\cdot, y) - a(\cdot, 0)) \nu(dy) \right\|_p \\ & \leq \int_{|y| \leq \varepsilon} |y| \left(\int_0^1 \|\nabla f(\cdot + sy) - \nabla f(\cdot)\|_p ds \right) \omega_a^{(0)}(|y|) \nu^{(\alpha)}(dy) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(7)}{\leq} C \|(-\Delta)^{\frac{\alpha-1}{2}} \nabla f\|_p \int_{|y| \leq \varepsilon} |y|^\alpha \omega_a^{(0)}(|y|) \nu^{(\alpha)}(dy) \\
&\leq C \|(-\Delta)^{\frac{\alpha}{2}} f\|_p \int_0^\varepsilon \frac{\omega_a^{(0)}(r)}{r} dr,
\end{aligned}$$

where the last step is due to (9) and the boundedness of Riesz transform in L^p -space. The case of $\alpha \in (0, 1)$ is similar. \square

For $a \in \mathcal{A}_0$, define the following nonlocal operator:

$$\mathcal{L}^{av} f(x) := \int_{\mathbb{R}^d} \mathcal{J}_f^{(\alpha)}(x, y) a(x, y) \nu(dy),$$

where $\mathcal{J}_f^{(\alpha)}(x, y)$ is given by (13). We now establish the following characterization about the domain of \mathcal{L}^{av} .

Theorem 3.4. *Let $\alpha \in (0, 2)$. Assume that $(\mathbf{H}_\nu^{(\alpha)})$ holds and $a \in \mathcal{A}_0$ satisfies that for some $0 < a_0 < a_1$ and any $0 < r < R < \infty$,*

$$a_0 \leq a(x, 0) \leq a_1, \quad 1_{\alpha=1} \int_{r \leq |y| \leq R} ya(x, y) \nu(dy) = 0. \quad (14)$$

Then for any $p \in (1, \infty)$, there exists a constant $C_1 \in (0, 1)$ depending only on $a_0, a_1, \nu_1^{(\alpha)}, \nu_2^{(\alpha)}$ and α, d, p such that for all $f \in \mathbb{H}^{\alpha, p}$,

$$C_1 \|f\|_{\alpha, p} \leq \|\mathcal{L}^{av} f\|_p + \|f\|_p \leq C_1^{-1} \|f\|_{\alpha, p}. \quad (15)$$

Proof. We make the following decomposition:

$$\begin{aligned}
\mathcal{L}^{av} f(x) &= a(x, 0) \mathcal{L}^\nu f(x) + \int_{|y| > \varepsilon} \mathcal{J}_f^{(\alpha)}(x, y) (a(x, y) - a(x, 0)) \nu(dy) \\
&\quad + \int_{|y| \leq \varepsilon} \mathcal{J}_f^{(\alpha)}(x, y) (a(x, y) - a(x, 0)) \nu(dy) \\
&=: I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

For $I_1(x)$, by Theorem 3.2 and condition (14), we have

$$a_0 C_0 \|(-\Delta)^{\alpha/2} f\|_p \leq \|I_1\|_p \leq a_1 C_0^{-1} \|(-\Delta)^{\alpha/2} f\|_p.$$

For $I_2(x)$, if $\alpha = 1$, by (14) we have

$$\|I_2\|_p = \left\| \int_{|y| > 1} [f(\cdot + y) - f(\cdot)] (a(\cdot, y) - a(\cdot, 0)) \nu(dy) \right\|_p \leq 4 \|f\|_p \|a\|_\infty \nu(B_1^c);$$

if $\alpha \in (0, 1)$, we have

$$\|I_2\|_p \leq 4 \|f\|_p \|a\|_\infty \nu(B_\varepsilon^c);$$

if $\alpha \in (1, 2)$, we have

$$\begin{aligned}
\|I_2\|_p &\leq 4 \|f\|_p \|a\|_\infty \int_{|y| > \varepsilon} \nu(dy) + 2 \|\nabla f\|_p \|a\|_\infty \int_{|y| > \varepsilon} |y| \nu(dy) \\
&\leq C_\varepsilon \|f\|_p + C_\varepsilon \|f\|_{\alpha, p}^{\frac{1}{\alpha}} \|f\|_p^{1 - \frac{1}{\alpha}} \leq \varepsilon \|f\|_{\alpha, p} + C_\varepsilon \|f\|_p.
\end{aligned}$$

For $I_3(x)$, by Lemma 3.3 we have

$$\|I_3\|_p \leq C \gamma(\varepsilon) \|(-\Delta)^{\alpha/2} f\|_p,$$

where $\gamma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Now, combining the above calculations, we obtain the right hand side estimate in (15). Moreover, we also have

$$\|\mathcal{L}^{\alpha\nu} f\|_p \geq \|I_1\|_p - \|I_2\|_p - \|I_3\|_p \geq (a_0 C_0 - \varepsilon - C\gamma(\varepsilon)) \|(-\Delta)^{\alpha/2} f\|_p - C_\varepsilon \|f\|_p.$$

Letting ε be small enough, we obtain the left hand side estimate in (15). \square

4. NONLOCAL LINEAR PARABOLIC EQUATION

In this section we fix a Lévy measure ν satisfying $(\mathbf{H}_\nu^{(\alpha)})$. Let $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonnegative and locally integrable function. Let $N(dt, dx)$ be the Poisson random point measure with intensity measure $\hat{N}(dt, dx) := \lambda(t)dt\nu(dx)$. Let $\tilde{N}(dt, dx) := N(dt, dx) - \hat{N}(dt, dx)$ be the compensated random martingale measure. Let $\vartheta : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ be a locally integrable function. For $t \geq 0$, define

$$X_t := \int_0^t \vartheta(r)dr + \int_0^t \int_{B^{(\alpha)}} y \tilde{N}(dr, dy) + \int_0^t \int_{\mathbb{R}^d - B^{(\alpha)}} y N(dr, dy), \quad (16)$$

where $B^{(\alpha)} = \{x : |x| \leq 1\}$ if $\alpha = 1$; $B^{(\alpha)} = \mathbb{R}^d$ if $\alpha \in (1, 2)$; and $B^{(\alpha)} = \emptyset$ if $\alpha \in (0, 1)$.

For $\varphi \in C_b^2(\mathbb{R}^d)$, by Itô's formula we have

$$\begin{aligned} \mathbb{E}\varphi(x + X_t - X_s) &= \varphi(x) + \mathbb{E} \int_s^t \vartheta(r) \cdot \nabla \varphi(x + X_r - X_s) dr \\ &+ \mathbb{E} \int_s^t \int_{\mathbb{R}^d} [\varphi(x + X_r - X_s + y) - \varphi(x + X_r - X_s) - y^{(\alpha)} \cdot \nabla \varphi(x + X_r - X_s)] \hat{N}(dr, dy). \end{aligned}$$

Thus, if we let

$$\mathcal{T}_{t,s}\varphi(x) := \mathcal{T}_{t,s}^{\lambda\nu,\vartheta}\varphi(x) := \mathbb{E}\varphi(x + X_t - X_s), \quad (17)$$

then one sees that

$$\partial_t \mathcal{T}_{t,s}\varphi = \mathcal{L}^{\lambda(t)\nu} \mathcal{T}_{t,s}\varphi + \vartheta(t) \cdot \nabla \mathcal{T}_{t,s}\varphi.$$

The following result is a slight extension of [19, Theorem 4.2].

Theorem 4.1. *Assume $(\mathbf{H}_\nu^{(\alpha)})$ with $\alpha \in (0, 2)$. Let $\vartheta : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ be a locally integrable function and $\lambda : \mathbb{R}_+ \rightarrow [\lambda_0, \infty)$ be a measurable function, where $\lambda_0 > 0$. Let $\mathcal{T}_{t,s}^{\lambda\nu,\vartheta}$ be defined by (17). Then for any $p \in (1, \infty)$, there exists a constant $C = C(\lambda_0, \nu_1^{(\alpha)}, \nu_2^{(\alpha)}, \alpha, p, d) > 0$ such that for any $T > 0$ and $f \in L^p((0, T) \times \mathbb{R}^d)$,*

$$\int_0^T \left\| \mathcal{L}^\nu \int_0^t \mathcal{T}_{t,s}^{\lambda\nu,\vartheta} f(s, \cdot) ds \right\|_p^p dt \leq C \int_0^T \|f(t)\|_p^p dt. \quad (18)$$

Proof. Let $N^{(1)}(dt, dx)$ and $N^{(2)}(dt, dx)$ be two independent Poisson random point measures with intensity measures $\hat{N}^{(1)}(dt, dx) := (\lambda(t) - \lambda_0)dt\nu(dx)$ and $\hat{N}^{(2)}(dt, dx) := \lambda_0 dt\nu(dx)$ respectively. Let $X_t^{(1)}$ be defined by (16) in terms of $N^{(1)}$, and $X_t^{(2)}$ be defined by

$$X_t^{(2)} := \int_0^t \int_{B^{(\alpha)}} y \tilde{N}^{(2)}(dr, dy) + \int_0^t \int_{\mathbb{R}^d - B^{(\alpha)}} y N^{(2)}(dr, dy).$$

In fact, $X_t^{(2)}$ is the Lévy process corresponding to the Lévy measure $\lambda_0\nu(dx)$. By Itô's formula, we have

$$\mathcal{T}_{t,s}^{\lambda\nu,\vartheta} f(x) = \mathbb{E}f(x + X_t^{(1)} - X_s^{(1)} + X_t^{(2)} - X_s^{(2)}) = \mathbb{E}\mathcal{T}_{t,s}^{\lambda_0\nu,0} f(x + X_t^{(1)} - X_s^{(1)}). \quad (19)$$

Thus, by [19, Theorem 4.2], we have

$$\begin{aligned} \int_0^T \left\| \mathcal{L}^\nu \int_0^t \mathcal{T}_{t,s}^{\lambda_0\nu,0} f(s, \cdot) ds \right\|_p^p dt &\leq \mathbb{E} \int_0^T \left\| \mathcal{L}^\nu \int_0^t \mathcal{T}_{t,s}^{\lambda_0\nu,0} f(s, \cdot + X_t^{(1)} - X_s^{(1)}) ds \right\|_p^p dt \\ &= \mathbb{E} \int_0^T \left\| \mathcal{L}^\nu \int_0^t \mathcal{T}_{t,s}^{\lambda_0\nu,0} f(s, \cdot - X_s^{(1)}) ds \right\|_p^p dt \\ &\leq C \mathbb{E} \int_0^T \|f(s, \cdot - X_s^{(1)})\|_p^p ds = C \int_0^T \|f(s)\|_p^p ds. \end{aligned}$$

The proof is complete. \square

Consider the following time-dependent linear nonlocal parabolic system:

$$\partial_t u = \mathcal{L}^{a(t)\nu} u + b^{(\alpha)} \cdot \nabla u + f, \quad u(0) = \varphi, \quad (20)$$

where $u, f : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^m$, $a : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are Borel measurable functions, and

$$b^{(\alpha)}(t, x) = 1_{\alpha \in [1,2)} b(t, x). \quad (21)$$

We make the following assumptions on a and b :

(H_v^a) For each $t \geq 0$, $a(t) \in \mathcal{A}_1$ satisfies

$$\sup_{t \in [0,1]} \|a(t)\|_{\mathcal{A}_1} < +\infty, \quad a_0 \leq a(t, x, 0) \leq a_1,$$

where $a_0, a_1 > 0$, and for all $0 < r < R < +\infty$,

$$1_{\alpha=1} \int_{r \leq |y| \leq R} ya(t, x, y) \nu(dy) = 0. \quad (22)$$

(H^b) For all $t \geq 0$ and $x, y \in \mathbb{R}^d$,

$$|b^{(\alpha)}(t, x) - b^{(\alpha)}(t, y)| \leq 1_{\alpha=1} \omega_b(|x - y|) + 1_{\alpha \in (1,2)} C_b,$$

where $\omega_b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing function with $\lim_{s \downarrow 0} \omega_b(s) = 0$.

Let us first prove the following apriori estimate by the method of freezing the coefficients (cf. [19, Lemma 5.1]).

Lemma 4.2. *Suppose that $a(t, x, y) = a(t, x)$ is independent of y and satisfies **(H_v^a)**, and b satisfies **(H^b)**. Let $p > 1$ and not equal to $\frac{\alpha}{\alpha-1}$ if $\alpha \in (1, 2)$, and let $f \in \mathbb{Y}^{0,p}$ and $u \in \mathbb{X}^{\alpha,p}$ satisfy (20). Then for all $t \in [0, 1]$,*

$$\|u\|_{\mathbb{X}_t^{\alpha,p}} \leq C \left(\|u(0)\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}} + \|f\|_{\mathbb{Y}_t^{0,p}} \right), \quad (23)$$

where C depends only on $a_0, a_1, \|a\|_{\mathcal{A}_1}, \|b\|_\infty, d, p, \alpha$ and ω_b .

Proof. Let $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$ be a family of mollifiers in \mathbb{R}^d , i.e., $\rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1}x)$, where $\rho \in C_0^\infty(\mathbb{R}^d)$ with $\int \rho = 1$ is nonnegative. Define

$$u_\varepsilon(t) := u(t) * \rho_\varepsilon, \quad a_\varepsilon(t) := a(t) * \rho_\varepsilon, \quad b_\varepsilon(t) := b(t) * \rho_\varepsilon, \quad f_\varepsilon(t) := f(t) * \rho_\varepsilon,$$

where $*$ stands for the convolution. Taking convolutions for both sides of (20), we have

$$\partial_t u_\varepsilon = \mathcal{L}^{a_\varepsilon(t)\nu} u_\varepsilon + b_\varepsilon^{(\alpha)} \cdot \nabla u_\varepsilon + h_\varepsilon, \quad (24)$$

where

$$h_\varepsilon := f_\varepsilon + (\mathcal{L}^{a(t)\nu} u) * \rho_\varepsilon - \mathcal{L}^{a_\varepsilon(t)\nu} u_\varepsilon + (b^{(\alpha)} \cdot \nabla u) * \rho_\varepsilon - b_\varepsilon^{(\alpha)} \cdot \nabla u_\varepsilon.$$

By the assumption, it is easy to see that for all $\varepsilon \in (0, 1)$ and $t \in [0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|a_\varepsilon(t, x) - a_\varepsilon(t, y)| \leq \omega_a^{(1)}(|x - y|), \quad |b_\varepsilon(t, x) - b_\varepsilon(t, y)| \leq 1_{\alpha=1}\omega_b(|x - y|) + 1_{\alpha \in (1,2)}C_b, \quad (25)$$

and

$$|a_\varepsilon(t, x) - a(t, x)| \leq \omega_a^{(1)}(\varepsilon), \quad |b_\varepsilon(t, x) - b(t, x)| \leq 1_{\alpha=1}\omega_b(\varepsilon) + 1_{\alpha \in (1,2)}C_b.$$

Moreover, by the property of convolutions, we also have

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 \|h_\varepsilon(t) - f(t)\|_p^p dt = 0.$$

Below, we use the method of freezing the coefficients to prove that for all $t \in [0, 1]$,

$$\|u_\varepsilon\|_{\mathbb{X}_t^{\alpha,p}} \leq C_{t,p} \left(\|u_\varepsilon(0)\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}} + \|h_\varepsilon\|_{\mathbb{Y}_t^{0,p}} \right), \quad (26)$$

where the constant C is independent of ε . After proving this estimate, (23) immediately follows by taking limits for (26).

For simplicity of notation, we drop the subscript ε below. Fix $\delta > 0$ being small enough, whose value will be determined below. Let ζ be a smooth function with support in B_δ and $\|\zeta\|_p = 1$. For $z \in \mathbb{R}^d$, set

$$\zeta_z(x) := \zeta(x - z), \quad \lambda_z^a(t) := a(t, z), \quad \vartheta_z^b(t) := 1_{\alpha=1}b(t, z).$$

Multiplying both sides of (24) by ζ_z , we have

$$\partial_t(u\zeta_z) = \lambda_z^a \mathcal{L}^\nu(u\zeta_z) + \vartheta_z^b \cdot \nabla(u\zeta_z) + g_z^\zeta,$$

where

$$g_z^\zeta := (a - \lambda_z^a) \mathcal{L}^\nu u \zeta_z + \lambda_z^a (\mathcal{L}^\nu u \zeta_z - \mathcal{L}^\nu(u\zeta_z)) + (b^{(\alpha)} - \vartheta_z^b) \cdot \nabla(u\zeta_z) - u b^{(\alpha)} \cdot \nabla \zeta_z + h \zeta_z.$$

Let $\mathcal{T}_{t,s}^{\lambda_z^a \nu, \vartheta_z^b}$ be defined by (17) in terms of $\lambda_z^a \nu$ and ϑ_z^b . By Duhamel's formula, $u\zeta_z$ can be written as

$$u\zeta_z(t, x) = \mathcal{T}_{t,0}^{\lambda_z^a \nu, \vartheta_z^b}(u(0)\zeta_z)(x) + \int_0^t \mathcal{T}_{t,s}^{\lambda_z^a \nu, \vartheta_z^b} g_z^\zeta(s, x) ds,$$

and so that for any $T \in [0, 1]$,

$$\begin{aligned} \int_0^T \|\mathcal{L}^\nu(u\zeta_z)(t)\|_p^p dt &\leq 2^{p-1} \left(\int_0^T \|\mathcal{L}^\nu \mathcal{T}_{t,0}^{\lambda_z^a \nu, \vartheta_z^b}(u(0)\zeta_z)\|_p^p dt + \int_0^T \left\| \mathcal{L}^\nu \int_0^t \mathcal{T}_{t,s}^{\lambda_z^a \nu, \vartheta_z^b} g_z^\zeta(s) ds \right\|_p^p dt \right) \\ &=: 2^{p-1} (I_1(T, z) + I_2(T, z)). \end{aligned}$$

For $I_1(T, z)$, by (19) and $\|\mathcal{L}^\nu f(\cdot + z)\|_p = \|\mathcal{L}^\nu f\|_p$, we have

$$\int_0^T \|\mathcal{L}^\nu \mathcal{T}_{t,0}^{\lambda_z^a \nu, \vartheta_z^b}(u(0)\zeta_z)\|_p^p dt = \int_0^T \|\mathcal{L}^\nu \mathcal{T}_{t,0}^{a_0 \nu, 0}(u(0)\zeta_z)\|_p^p dt \leq C \|u(0)\zeta_z\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}}^p, \quad (27)$$

where the last step is due to [17, p.96 Theorem 1.14.5] and [19, Corollary 4.5]. Thus, by definition (5), it is easy to see that

$$\int_{\mathbb{R}^d} I_1(T, z) dz \leq C \int_{\mathbb{R}^d} \|u(0)\zeta_z\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}}^p dz \leq C \left(\|u(0)\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}}^p \|\zeta\|_p^p + \|u(0)\|_p^p \|\zeta\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}}^p \right).$$

For $I_2(T, z)$, by Theorem 4.1, we have

$$I_2(T, z) \leq C \int_0^T \|g_z^\zeta(s)\|_p^p ds \leq C \int_0^T \|((a - \lambda_z^a)(\mathcal{L}^\nu u \zeta_z))(s)\|_p^p ds$$

$$\begin{aligned}
& + C \int_0^T \|\lambda_z^a(\mathcal{L}^\nu(u\zeta_z) - \mathcal{L}^\nu u\zeta_z)(s)\|_p^p ds \\
& + C \int_0^T \|((b^{(\alpha)} - \vartheta_z^b) \cdot \nabla(u\zeta_z))(s)\|_p^p ds \\
& + C \int_0^T \|(ub^{(\alpha)} \cdot \nabla\zeta_z)(s)\|_p^p ds + C \int_0^T \|h\zeta_z(s)\|_p^p ds \\
& =: I_{21}(T, z) + I_{22}(T, z) + I_{23}(T, z) + I_{24}(T, z) + I_{25}(T, z).
\end{aligned}$$

For $I_{21}(T, z)$, by (25) and $\|\zeta\|_p = 1$, we have

$$\int_{\mathbb{R}^d} I_{21}(T, z) dz \leq C\omega_a^{(1)}(\delta)^p \int_{\mathbb{R}^d} \int_0^T \|(\mathcal{L}^\nu u\zeta_z)(s)\|_p^p ds dz = C\omega_a^{(1)}(\delta)^p \int_0^T \|\mathcal{L}^\nu u(s)\|_p^p ds.$$

For $I_{22}(T, z)$, using (7) and as in the proof of [19, Lemma 2.5], for any $\beta \in (0 \vee (\alpha - 1), \alpha)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} I_{22}(T, z) dz & \leq Ca_1 \int_0^T \int_{\mathbb{R}^d} \|(\mathcal{L}^\nu(u\zeta_z) - \mathcal{L}^\nu u\zeta_z)(s)\|_p^p dz ds \\
& \leq C \int_0^T \|u(s)\|_p^p ds + C \int_0^T \|(-\Delta)^{\beta/2} u(s)\|_p^p ds \\
& \leq C \int_0^T \|u(s)\|_p^p ds + \frac{1}{4^p} \int_0^T \|\mathcal{L}^\nu u(s)\|_p^p ds,
\end{aligned}$$

where the last step is due to the interpolation inequality, Young's inequalities and Theorem 3.2.

For $I_{23}(T, z)$, as above we have

$$\int_{\mathbb{R}^d} I_{23}(T, z) dz \leq C1_{\alpha=1}\omega_b(\delta)^p \left(\int_0^T \|\nabla u(s)\|_p^p ds + \|\nabla \zeta\|_p^p \int_0^T \|u(s)\|_p^p ds \right).$$

Moreover, it is easy to see that

$$\begin{aligned}
\int_{\mathbb{R}^d} I_{24}(T, z) dz & \leq C\|b\|_\infty^p \|\nabla \zeta\|_p^p \int_0^T \|u(s)\|_p^p ds, \\
\int_{\mathbb{R}^d} I_{25}(T, z) dz & \leq C \int_0^T \|h(s)\|_p^p ds.
\end{aligned}$$

Combining the above calculations, we get

$$\begin{aligned}
\int_0^T \|\mathcal{L}^\nu u(s)\|_p^p ds & = \int_0^T \int_{\mathbb{R}^d} \|\mathcal{L}^\nu u(s) \cdot \zeta_z\|_p^p dz ds \leq 2^{p-1} \int_0^T \int_{\mathbb{R}^d} \|\mathcal{L}^\nu(u\zeta_z)(s)\|_p^p dz ds \\
& + 2^{p-1} \int_0^T \int_{\mathbb{R}^d} \|(\mathcal{L}^\nu u\zeta_z - \mathcal{L}^\nu(u\zeta_z))(s)\|_p^p dz ds \\
& \leq C\|u(0)\|_{\mathbb{W}^{\alpha-\frac{q}{p}, p}}^p + \left(\frac{1}{4} + C(\omega_a^{(1)}(\delta)^p + \omega_b(\delta)^p) \right) \int_0^T \|\mathcal{L}^\nu u(s)\|_p^p ds \\
& + C \int_0^T \|u(s)\|_p^p ds + C \int_0^T \|h(s)\|_p^p ds.
\end{aligned}$$

Choosing $\delta_0 > 0$ being small enough so that

$$C(\omega_a^{(1)}(\delta_0)^p + \omega_b(\delta_0)^p) \leq \frac{1}{4},$$

we obtain that for all $T \in [0, 1]$,

$$\int_0^T \|\mathcal{L}^\nu u(s)\|_p^p ds \leq C \|u(0)\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}}^p + C \int_0^T \|u(s)\|_p^p ds + C \int_0^T \|h(s)\|_p^p ds. \quad (28)$$

On the other hand, by (24), it is easy to see that for all $t \in [0, 1]$,

$$\|u(t)\|_p^p \leq C \|u(0)\|_p^p + C 1_{\alpha \in [1,2)} \int_0^t \|\nabla u(s)\|_p^p ds + C \int_0^t \|h(s)\|_p^p ds,$$

which together with (28) and Gronwall's inequality yields that for all $t \in [0, 1]$,

$$\sup_{s \in [0,t]} \|u(s)\|_p^p + \int_0^t \|\mathcal{L}^\nu u(s)\|_p^p ds \leq C \left(\|u(0)\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}}^p + \int_0^t \|h(s)\|_p^p ds \right). \quad (29)$$

From equation (24), we also have

$$\int_0^t \|\partial_s u(s)\|_p^p ds \leq C \left(\|a\|_\infty^p \int_0^t \|\mathcal{L}^\nu u(s)\|_p^p ds + \|b^{(\alpha)}\|_\infty^p \int_0^t \|\nabla u(s)\|_p^p ds + \int_0^t \|h(s)\|_p^p ds \right),$$

which together with (29) and (12) gives (26), and therefore (23). \square

We now prove the following main result of this paper.

Theorem 4.3. *Suppose $(\mathbf{H}_\nu^{(\alpha)})$, (\mathbf{H}_ν^a) and (\mathbf{H}^b) and for some $k \in \mathbb{N} \cup \{0\}$,*

$$|\nabla_x^j a(t, x, y)| + |\nabla_x^j b(t, x)| \leq C_j, \quad j = 0, \dots, k.$$

For given $p \in (1, \infty)$ not equal to $\frac{\alpha}{\alpha-1}$ when $\alpha \in (1, 2)$, and for $\varphi \in \mathbb{W}^{k+\alpha-\frac{\alpha}{p},p}$, there exists a unique $u \in \mathbb{X}^{k+\alpha,p}$ satisfying equation (20). Moreover, for all $t \in [0, 1]$,

$$\|u\|_{\mathbb{X}_t^{k+\alpha,p}} \leq C_{k,p} \left(\|\varphi\|_{\mathbb{W}^{k+\alpha-\frac{\alpha}{p},p}} + \|f\|_{\mathbb{Y}_t^{k,p}} \right), \quad (30)$$

where $C_{0,p}$ depends only on $a_0, a_1, \|a\|_{\mathcal{A}_1}, \|b\|_\infty, d, p, \alpha$ and ω_b .

Proof. The strategy is to prove the apriori estimate (30) and then use the continuity method (cf. [12, 20]).

(Step 1) Let us first rewrite equation (20) as

$$\partial_t u(t, x) = a(t, x, 0) \mathcal{L}^\nu u(t, x) + b^{(\alpha)}(t, x) \cdot \nabla u(t, x) + \tilde{f}(t, x),$$

where

$$\tilde{f}(t, x) := f(t, x) + \int_{\mathbb{R}^d} \mathcal{J}_{u(t,\cdot)}^{(\alpha)}(x, y) (a(t, x, y) - a(t, x, 0)) \nu(dy)$$

and

$$\mathcal{J}_{u(t,\cdot)}^{(\alpha)}(x, y) := u(t, x+y) - u(t, x) - y^{(\alpha)} \cdot \nabla u(t, x).$$

Notice that by Lemma 3.3,

$$\begin{aligned} \|\tilde{f}(t)\|_p &\leq \|f(t)\|_p + \left\| \int_{|y|>\varepsilon} \mathcal{J}_{u(t,\cdot)}^{(\alpha)}(\cdot, y) (a(t, \cdot, y) - a(t, \cdot, 0)) \nu(dy) \right\|_p \\ &\quad + \left\| \int_{|y|\leq\varepsilon} \mathcal{J}_{u(t,\cdot)}^{(\alpha)}(\cdot, y) (a(t, \cdot, y) - a(t, \cdot, 0)) \nu(dy) \right\|_p \\ &\leq \|f(t)\|_p + 2a_1 \left(\|u(t)\|_p \nu(B_\varepsilon^c) + 1_{\alpha \in (1,2)} \|\nabla u(t)\|_p \right) \\ &\quad + C_\varepsilon \|u(t)\|_p + C\gamma_0(\varepsilon) \|(-\Delta)^{\frac{\alpha}{2}} u(t)\|_p \\ &\leq \|f(t)\|_p + C_\varepsilon \|u(t)\|_p + \gamma_1(\varepsilon) \|(-\Delta)^{\frac{\alpha}{2}} u(t)\|_p, \end{aligned}$$

where the last step is due to the interpolation inequality and Young's inequalities, and

$$\gamma_0(\varepsilon) := \int_0^\varepsilon \frac{\omega_a^{(0)}(r)}{r} dr, \quad \gamma_1(\varepsilon) := \varepsilon + C\gamma_0(\varepsilon).$$

By Lemma 4.2, we have

$$\|u\|_{\mathbb{X}_t^{\alpha,p}} \leq C_1 \|\varphi\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}} + C_2 \|\tilde{f}\|_{\mathbb{Y}_t^{0,p}}.$$

In particular, for all $t \in [0, 1]$,

$$\begin{aligned} \sup_{s \in [0,t]} \|u(s)\|_p + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|_p^p ds &\leq C_1 \|\varphi\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}} + \gamma_1(\varepsilon) \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|_p^p ds \\ &\quad + C_2 \int_0^t \|u(s)\|_p^p ds + C_2 \int_0^t \|f(s)\|_p^p ds. \end{aligned}$$

Letting ε be small enough and using Gronwall's inequality, we obtain (30) with $k = 0$.

(Step 2) We now estimate the higher order derivatives. Write

$$w^{(n)}(t, x) := \nabla^n u(t, x).$$

By the chain rule, one can see that

$$\partial_t w^{(n)} = \mathcal{L}^{a\nu} w^{(n)} + b^{(\alpha)} \cdot \nabla w^{(n)} + g^{(n)},$$

where

$$g^{(n)} := \nabla^n f + \sum_{j=1}^n \frac{n!}{(n-j)!j!} \left(\mathcal{L}^{(\nabla_x^j a)\nu} (\nabla^{n-j} u) + \nabla^j b^{(\alpha)} \cdot \nabla^{n-j+1} u \right)$$

and

$$\mathcal{L}^{(\nabla_x^j a)\nu} (\nabla^{n-j} u)(t, x) = \int_{\mathbb{R}^d} \mathcal{J}_{\nabla^{n-j} u(t, \cdot)}^{(\alpha)}(x, y) \nabla_x^j a(t, x, y) \nu(dy).$$

By Step 1, we know that

$$\|w^{(n)}\|_{\mathbb{X}_t^{\alpha,p}} \leq C \left(\|w^{(n)}(0)\|_{\mathbb{W}^{\alpha-\frac{\alpha}{p},p}} + \|g^{(n)}\|_{\mathbb{Y}_t^{0,p}} \right). \quad (31)$$

By Minkowskii's inequality, we have

$$\begin{aligned} \|\mathcal{L}^{(\nabla_x^j a)\nu} (\nabla^{n-j} u)(t)\|_p &\leq C_j \int_{\mathbb{R}^d} \|\nabla^{n-j} u(t, \cdot + y) - \nabla^{n-j} u(t, \cdot) - y^{(\alpha)} \cdot \nabla \nabla^{n-j} u(t, \cdot)\|_p \nu(dy) \\ &\leq C_j \int_{|y| \geq 1} \left(2\|\nabla^{n-j} u(t)\|_p + 1_{\alpha \in (1,2)} |y| \cdot \|\nabla^{n-j+1} u(t)\|_p \right) \nu(dy) \\ &\quad + C_j 1_{\alpha \in (0,1)} \int_{|y| \leq 1} \|\nabla^{n-j} u(t, \cdot + y) - \nabla^{n-j} u(t, \cdot)\|_p \nu(dy) \\ &\quad + C_j 1_{\alpha \in [1,2)} \int_{|y| \leq 1} \|\nabla^{n-j} u(t, \cdot + y) - \nabla^{n-j} u(t, \cdot) - y \cdot \nabla \nabla^{n-j} u(t, \cdot)\|_p \nu(dy) \\ &\leq 2C_j \nu(B_1^c) \|\nabla^{n-j} u(t)\|_p + C_j \|\nabla^{n-j+1} u(t)\|_p \\ &\quad \times \left(\int_{B_1^c} |y| 1_{\alpha \in (1,2)} \nu(dy) + \int_{B_1} |y| 1_{\alpha \in (0,1)} \nu(dy) \right) \\ &\quad + C_j 1_{\alpha \in [1,2)} \int_{|y| \leq 1} |y| \left(\int_0^1 \|\nabla^{n-j+1} u(t, \cdot + sy) - \nabla^{n-j+1} u(t, \cdot)\|_p ds \right) \nu(dy) \\ &\leq C \|\nabla^{n-j} u(t)\|_p + C \|\nabla^{n-j+1} u(t)\|_p + C \|\nabla^{n-j+1} u(t)\|_{\beta,p} \int_{B_1} |y|^{1+\beta} 1_{\alpha \in [1,2)} \nu(dy), \end{aligned}$$

where $\beta \in ((\alpha - 1) \vee 0, 1)$ and the last step is due to (7).

Hence, by the assumptions, we obtain

$$\|g^{(n)}\|_{\mathbb{Y}_t^{0,p}}^p \leq \|f\|_{\mathbb{Y}_t^{n,p}}^p + C\|u\|_{\mathbb{Y}_t^{n,p}}^p + C_t\|u\|_{\mathbb{Y}_t^{n+\beta,p}}^p 1_{\alpha \in [1,2)}.$$

Summing over n from 0 to k for (31) yields

$$\begin{aligned} \|u(t)\|_{\mathbb{W}^{k,p}}^p + \int_0^t \|u(s)\|_{\mathbb{W}^{k+\alpha,p}}^p ds &\leq C\|\varphi\|_{\mathbb{W}^{k+\alpha-\frac{\alpha}{p},p}}^p + C1_{\alpha \in [1,2)} \int_0^t \|u(s)\|_{\mathbb{W}^{k+\beta,p}}^p ds \\ &\quad + C \int_0^t \|f(s)\|_{\mathbb{W}^{k,p}}^p ds + C \int_0^t \|u(s)\|_{\mathbb{W}^{k,p}}^p ds \\ &\leq C\|\varphi\|_{\mathbb{W}^{k+\alpha-\frac{\alpha}{p},p}}^p + C1_{\alpha \in [1,2)} \int_0^t \|u(s)\|_{\mathbb{W}^{k+\alpha,p}}^{p\beta/\alpha} \|u(s)\|_{\mathbb{W}^{k,p}}^{p(1-\beta/\alpha)} ds \\ &\quad + C \int_0^t \|f(s)\|_{\mathbb{W}^{k,p}}^p ds + C \int_0^t \|u(s)\|_{\mathbb{W}^{k,p}}^p ds \\ &\leq C\|\varphi\|_{\mathbb{W}^{k+\alpha-\frac{\alpha}{p},p}}^p + \frac{1}{2}1_{\alpha \in [1,2)} \int_0^t \|u(s)\|_{\mathbb{W}^{k+\alpha,p}}^p ds \\ &\quad + C \int_0^t \|f(s)\|_{\mathbb{W}^{k,p}}^p ds + C \int_0^t \|u(s)\|_{\mathbb{W}^{k,p}}^p ds, \end{aligned}$$

which then gives (30) by Gronwall's inequality.

(Step 3) For $\lambda \in [0, 1]$, define an operator

$$U_\lambda := \partial_t - \lambda \mathcal{L}^{av} - \lambda b^{(\alpha)} \cdot \nabla - (1 - \lambda) \mathcal{L}^\nu.$$

By (15), it is easy to see that

$$U_\lambda : \mathbb{X}^{k+\alpha,p} \rightarrow \mathbb{Y}^{k,p}. \quad (32)$$

For given $\varphi \in \mathbb{W}^{k+\alpha-\frac{\alpha}{p},p}$, let $\mathbb{X}_\varphi^{k+\alpha,p}$ be the space of all functions $u \in \mathbb{X}^{k+\alpha,p}$ with $u(0) = \varphi$. It is clear that $\mathbb{X}_\varphi^{k+\alpha,p}$ is a complete metric space with respect to the metric $\|\cdot\|_{\mathbb{X}^{k+\alpha,p}}$. For $\lambda = 0$ and $f \in \mathbb{Y}^{k,p}$, it is well-known that there is a unique $u \in \mathbb{X}_\varphi^{k+\alpha,p}$ such that

$$U_0 u = \partial_t u - \mathcal{L}^\nu u = f.$$

In fact, by Duhamel's formula, the unique solution can be represented by

$$u(t, x) = \mathcal{T}_{t,0}^{\nu,0} \varphi(x) + \int_0^t \mathcal{T}_{t,s}^{\nu,0} f(s, x) ds,$$

where $\mathcal{T}_{t,s}^{\nu,0}$ is defined by (17). Suppose now that for some $\lambda_0 \in [0, 1]$, and for any $f \in \mathbb{Y}^{k,p}$, the equation

$$U_{\lambda_0} u = f$$

admits a unique solution $u \in \mathbb{X}_\varphi^{k+\alpha,p}$. Thus, for fixed $f \in \mathbb{Y}^{k,p}$ and $\lambda \in [\lambda_0, 1]$, and for any $u \in \mathbb{X}_\varphi^{k+\alpha,p}$, by (32), the equation

$$U_{\lambda_0} w = f + (U_{\lambda_0} - U_\lambda)u \quad (33)$$

admits a unique solution $w \in \mathbb{X}_\varphi^{k+\alpha,p}$. Introduce an operator

$$w = Q_\lambda u.$$

We now use the apriori estimate (30) to show that there exists an $\varepsilon > 0$ independent of λ_0 such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$,

$$Q_\lambda : \mathbb{X}_\varphi^{k+\alpha,p} \rightarrow \mathbb{X}_\varphi^{k+\alpha,p}$$

is a contraction operator.

Let $u_1, u_2 \in \mathbb{X}_\varphi^{k+\alpha,p}$ and $w_i = Q_\lambda u_i, i = 1, 2$. By equation (33), we have

$$U_{\lambda_0}(w_1 - w_2) = (U_{\lambda_0} - U_\lambda)(u_1 - u_2) = (\lambda_0 - \lambda)(\mathcal{L}^{(a-1)\nu} + b^{(\alpha)} \cdot \nabla)(u_1 - u_2).$$

By (30) and (15), it is not hard to see that

$$\begin{aligned} \|Q_\lambda u_1 - Q_\lambda u_2\|_{\mathbb{X}^{k+\alpha,p}} &\leq C_{k,p}|\lambda_0 - \lambda| \cdot \|(\mathcal{L}^{(a-1)\nu} + b^{(\alpha)} \cdot \nabla)(u_1 - u_2)\|_{\mathbb{Y}^{k,p}} \\ &\leq C_0|\lambda_0 - \lambda| \cdot \|u_1 - u_2\|_{\mathbb{X}^{k+\alpha,p}}, \end{aligned}$$

where C_0 is independent of λ, λ_0 and u_1, u_2 . Taking $\varepsilon = 1/(2C_0)$, one sees that

$$Q_\lambda : \mathbb{X}_\varphi^{k+\alpha,p} \rightarrow \mathbb{X}_\varphi^{k+\alpha,p}$$

is a $1/2$ -contraction operator. By the fixed point theorem, for each $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, there exists a unique $u \in \mathbb{X}_\varphi^{k+\alpha,p}$ such that

$$Q_\lambda u = u,$$

which means that

$$U_\lambda u = f.$$

Now starting from $\lambda = 0$, after repeating the above construction $[\frac{1}{\varepsilon}] + 1$ -steps, one obtains that for any $f \in \mathbb{Y}^{k,p}$,

$$U_1 u = f$$

admits a unique solution $u \in \mathbb{X}_\varphi^{k+\alpha,p}$. □

REFERENCES

- [1] Amann H.: Linear and quasilinear parabolic problems. Vol. I, Abstract linear theory. Monographs in Mathematics, Vol.89, Birkhäuser Boston, MA, 1995.
- [2] Applebaum D.: Lévy processes and stochastic calculus. Cambridge Studies in Advance Mathematics 93, Cambridge University Press, 2004.
- [3] Barles G., Chasseigne E. and Imbert C.: Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. J. Eur. Math. Soc., Vol.13, 1-26(2011).
- [4] Bergh J. and Löfström J.: Interpolation spaces. Grundlehren der math. Wissen., 223, Springer-Verlag, 1976.
- [5] Biler P., Funaki T., Woyczynski W.A.: Fractal Burgers equations. J. Diff. Equa., 148, 9-46(1998).
- [6] Caffarelli L., Chan C.H. and Vasseur A.: Regularity theory for nonlinear integral operators. J. Amer. Math. Soc. 24 (2011) 849-869.
- [7] Caffarelli L. and Vasseur A.: Drift diffusion equations with fractional diffusion and the quasigeostrophic equation. Annals of Math., Vol. 171, No. 3, 1903-1930(2010).
- [8] Dong H. and Kim D.: On L_p -estimates for a class of nonlocal elliptic equations. arXiv:1102.4073v1.
- [9] Gilboa G. and Osher S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul., 7(3):1005-1028, 2008.
- [10] Pazy A.: Semigroups of linear operators and applications to partial differential equations. Applied Mathematics Sciences, vol.44, Springer-Verlag, 1983.
- [11] Komatsu T.: On the martingale problem for generators of stable processes with perturbations. Osaka J. Math. 21(1984),113-132.
- [12] Krylov N.V.: Lectures on Elliptic and Parabolic Equations in Sobolev Spaces. AMS, Graduate Studies in Mathematics, Vol. 96, 2008.
- [13] Mikulevicius R. and Pragarauskas H.: On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces. Lithuanian Math. Journal, Vol.32, No.2, 1992.
- [14] Sato, K.: Lévy processes and infinitely divisible distributions. Cambridge University Press, 1999.

- [15] Stein E.M.: Singular integrals and differentiability properties of functions. Princeton, N.J., Princeton University Press, 1970.
- [16] Stein E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, 1993.
- [17] Triebel H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland Publishing Company, Amsterdam, 1978.
- [18] Zhang X.: Stochastic Differential Equations with Sobolev Drifts and Driven by α -stable Processes. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, in press.
- [19] Zhang X.: L^p -maximal regularity of nonlocal parabolic equation and applications. <http://arxiv.org/abs/1109.0816>.
- [20] Zhang X.: Well-posedness of fully nonlinear and nonlocal critical parabolic equations. <http://arxiv.org/abs/1111.1874>.

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